

# A search for the minimal unified field theory. II. Spinor matter and gravity.

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The spin connections of the Dirac field have three ingredients that are connected with the Ricci rotations, the Maxwell field, and an *axial field* which is coupled to the axial current. I demonstrate that the axial field provides an effective mechanism of auto-localization of the Dirac field into compact objects. A non-linear system of Dirac and sine-Gordon equations that has a potential to yield a mass spectrum is derived. The condition that the compact objects are stable (the energy-momentum is self-adjoint) leads to the Einstein's field equations. The Dirac field with its spin connection seem to be a natural material carrier of the space-time continuum in which the compact objects are moving along the geodesic lines. The long distance effect of the axial field is indistinguishable from the Newton's gravity which reveals the microscopic nature of gravity and the origin of the gravitational mass.

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## I. INTRODUCTION

In this paper, I continue to explore an hypothesis that the Dirac spinor field is the primary form of matter and that its dynamics is driven exclusively by its spin connections. In the previous paper [1], it was shown that a residual degree of freedom, which is left to the Dirac spinor field by the rule of the Lorentz transformation of its probability current, allows for the existence of *two* real vector fields. One of these fields,  $A^a(x)$ , has the properties of the Maxwell field derived within the standard scheme of gauge invariance [2, 3, 4]. This field is minimally coupled to the conserved probability current  $j^a$ . The second field, the axial field  $\aleph^a(x)$ , is minimally coupled to the axial current  $J_5^a \equiv \mathcal{J}^a$  which, in turn, has a pseudoscalar density  $\mathcal{P}$  as a source. This current is not conserved and  $\aleph^a$  is a massive (not gradient invariant) field. The vector field  $A^a$  affects the parallel transport of a four-component Dirac spinor *as a whole*, while the  $\aleph^a$  field acts on its left and right components differently. Therefore, the axial field becomes most visible in the *polarization* characteristics of Dirac spinors, i.e. bilinear forms with the left- and right- spinor components mixed.

In the context of this study, both fields,  $A^a$  and  $\aleph^a$ , are derived and treated as the spin connections of the Dirac field  $\psi$ . The spinors themselves represent a local (and only local) group of the Lorentz transformations and the probability current of two-component spinors is light-like. Therefore, the spinor fields do not set any limit on the space-time resolution in all conceivable measurements. The dynamics of spinors dominate all physical phenomena where sharp space-time localization occurs. Since the light-like propagation is an intrinsic property of spinor fields they provide a material footing for the first axiom of special relativity – all signals must travel at the speed of light. The spin connections naturally inherit

this property.

Special relativity also requires (the second axiom) the existence of inertial observers that have finite sizes and lifetimes so that the notion of their proper time is sensible. In scattering theory, it is closely related to the so-called *cluster decomposition principle*. Construction of such finite-sized objects is a long-standing problem of the relativistic field theory. In this paper, I show that a modest and natural requirement that particles are defined as some configurations of the Dirac field that do not change for a period of time (are parallel-transported) makes the spin connection  $\aleph^a$  the *gradient of a scalar function*. Then  $\aleph^a$  becomes a force that leads to tightly bound configurations of the spinor field. This effect is so pronounced that one may think of auto-localization as one of the genuine properties of the Dirac field. In other words, the Dirac field, with its spin connections, provides a material support to the whole physical structure, which is called *the space-time continuum*. The gravitational interaction is initiated by the axial current and is transmitted by the (static or propagating) axial field and axial polarization; at large distances, it is indistinguishable from Newton's gravity.

The objective of this paper is to show that a theory based on the interaction of only three fields,  $\psi$ ,  $A^\mu$  and  $\aleph^\mu$  is self-consistent. The main result of this work is contained in Eqs. (3.9) and (3.11), given below for convenience,

$$\alpha^a \{ \partial_a + ieA_a(x) + (i/2)\rho_3 \partial_a \Upsilon(x) - \Omega_a(x) \} \psi + im\rho_1 \psi = 0, \quad (\text{I.1})$$

$$\square \Upsilon(x) - \frac{4g^2 m}{M^2} \mathcal{R}(x) \sin \Upsilon(x) = 0, \quad \mathcal{R}^2 = (\psi^+ \alpha^a \psi)^2. \quad (\text{I.2})$$

This non-linear system of equations is capable of describing the interaction and spectrum of masses/scales/ energies of localized states at short distances as well as propagation of these states as compact objects. The first of

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these equations is the linear Dirac equation which is modified by the presence of an axial field. It can be considered as a microscopic equation that preserves the superposition principle. The second equation interpolates between the short and long distances in a sense that it describes the axial field subjected to the condition of parallel transport of a compact spinor object. At large distances, the field  $\Upsilon(x)$  becomes the Newton potential; it is responsible for a residual (gravitational) interaction between Dirac clusters.

An additional condition that the spinor clusters remain stable (the Dirac operator and the operator of energy-momentum are self-adjoint) yields Eqs. (4.8) and (4.4),

$$R_{\lambda\sigma} = 0, \quad (\text{I.3})$$

$$\Gamma_{\mu\nu}^\sigma T_\sigma^\nu = -m\mathcal{P} \partial_\mu \Upsilon, \quad (\text{I.4})$$

where  $R_{\lambda\sigma}$  is the Ricci curvature tensor. The first of these is the Einstein equation for the metric of free space. The second is a manifestation of local equivalence between the axial-Newton field and the forces of inertia. Eq. (I.4) is the bridge between the physical dynamics of spinor fields and the origin of dynamical metrics of general relativity.

There appears to be no problem of *including* the spinor fields *into* general relativity. The equations of general relativity emerge as one of the constituents of spinor dynamics in the limit when the spinor fields and their spin connections form particles that may serve as rods and clocks, i.e., the inertial frames which, in their turn, are necessary to verify the initial hypothesis of local Lorentz invariance. In agreement with the Einstein's eventual judgement [5], the field equation (I.3) has no energy-momentum tensor of matter in its right-hand side. The pseudoscalar  $\mathcal{P}$ , which is the maximum of the probability density localized in a compact spinor object, takes the role of gravitational mass. *The physical origin of the macroscopic forces of gravity between any two bodies is a trend of the global Dirac field to concentrate around the microscopic domains where this field happened to be extremely localized.* These forces tend to polarize matter at the level of its spinor organization and may well play a role at various stages of matter evolution.

The paper is organized as follows: Sec. II is a review of the results of the previous work [1], which are modified by incorporating an explicit requirement that the Dirac operator be self-adjoint. In Sec. III, I formulate a criterion that the Dirac field represents a compact object. The ansatz of parallel transport of Dirac particles is introduced, and the form of the spin connection, compatible with this ansatz, is found. In Sec. IV, I show that the integrity of a Dirac particle implies that it moves along a geodesic world line of the metric background that satisfies Einstein's field equation. A model that demonstrates the auto-localization in a simple spherical geometry is also worked out in Sec. V.

## II. PARALLEL TRANSPORT OF THE SPINOR FIELD

The affine connection in the covariant derivative of a vector field can be derived in a relatively simple way because rotation of a vector at a given point follows the rotation of the *local* coordinate axes. Since the Lorentz spinors are defined locally and only locally and their components are not given in terms of tensor variables, there is no similar rule for spinor fields. One has to resort to the so-called tetrad formalism [6], which conforms to the principle of the equivalence of local inertial frames in special relativity.

The curvilinear coordinates, which are used throughout this study, are not connected *a priori* with the true curvature of space-time. The concept of a compact spinor object naturally leads to the systems of coordinates in which at least one "radial" direction is parameterized by closed two-dimensional surfaces. Even in the absence of a true space-time curvature, the affine and spin connections remain indispensable attributes of this scheme. For a stable compact object, the transport along closed surfaces can have a group property.

### A. Spin connection

The most important physical quantity that provides access to the geometric properties of spinors is the probability current. Its time component is a unit operator that commutes with everything. Thus, it corresponds to the most fundamental measurement of quantum mechanics. This current,  $j_a = \psi^\dagger \alpha_a \psi \equiv \psi^\dagger (1, \rho_3 \sigma_i) \psi \equiv \bar{\psi} \gamma_a \psi$ , must be a Lorentz vector, which is transformed as  $j_a(x) \rightarrow \Lambda_a^b(x) j_b(x)$  under a local Lorentz rotation, and its variation under the parallel displacement  $dx^\mu$  is  $\delta j_\mu = \Gamma_{\mu\lambda}^\nu j_\nu dx^\lambda$ . The tetrad components of this vector change by  $\delta j_a = \omega_{acb} j^c ds^b = \omega_{acb} \psi^\dagger \alpha^c \psi ds^b$  when this vector is transported by  $ds^a$ . In these equations,  $\Gamma_{\mu\lambda}^\nu$  are the Christoffel symbols,  $\partial_a = e_a^\mu \partial_\mu$  is the derivative in direction  $a$ , and  $\omega_{abc}$  are the Ricci rotation coefficients, ( $\omega_{bca} = (\nabla_\mu e_b^\nu) e_{c\nu} e_a^\mu = -\omega_{cba}$ ).

The convention for Dirac matrices is as follows: The basic matrices  $\rho_i$  and  $\sigma_i$ ,  $i = 1, 2, 3$ , were introduced by Dirac [7]; we use  $\rho_0 = \sigma_0 = \mathbf{1}$ , for unit matrix. The other notations are:  $\alpha_a = (\alpha_0, \alpha_i)$  (with  $\alpha_0 = 1$ ,  $\alpha_i = \rho_3 \sigma_i$ ),  $\rho_1 = \beta = \gamma^0$ ,  $\rho_2 = -i\gamma^0 \gamma^5$ , and  $\rho_3 = -\gamma^5$ . The  $4 \times 4$  matrices  $\sigma$  and  $\rho$  satisfy the same commutation relations as the Pauli matrices, and all matrices  $\sigma$  commute with all matrices  $\rho$ , i.e.,  $\sigma_i \sigma_k = \delta_{ik} + i\epsilon_{ikl} \sigma_l$ ,  $\rho_a \rho_b = \delta_{ab} + i\epsilon_{abc} \rho_c$ , and  $\sigma_i \rho_a - \rho_a \sigma_i = 0$ .

Let matrix  $\Gamma_a$  (the spin connection) define the change of the spinor components in the course of the same infinitesimal displacement,  $\delta\psi = \Gamma_a \psi ds^a$ ,  $\delta\psi^\dagger = \psi^\dagger \Gamma_a^\dagger ds^a$ . This gives yet another expression for  $\delta j_a$ , namely,  $\delta j_a = \psi^\dagger (\Gamma_b^\dagger \alpha_a + \alpha_a \Gamma_b) \psi ds^b$ . The two forms of  $\delta j_a$  must be the same. Hence, the equation that de-

fines  $\Gamma_a$  is

$$\Gamma_b^+ \alpha_a + \alpha_a \Gamma_b = \omega_{acb} \alpha^c, \quad (2.1)$$

and it has been shown to have the most general solution [1],

$$\Gamma_b(x) = -ieA_b(x) - ig\rho_3 \aleph_b(x) + \Omega_b(x), \quad (2.2)$$

where the last term is the geometric part,

$$\Omega_b(x) = \frac{1}{4} \omega_{cdb}(x) \rho_1 \alpha^c \rho_1 \alpha^d,$$

and the signs of coupling constants are chosen with the electron in mind. The covariant derivative of a spinor now reads as

$$D_a \psi = (\partial_a - \Gamma_a) \psi, \quad \Gamma_\mu = e_\mu^a \Gamma_a,$$

$$D_\mu \psi = e_\mu^a D_a \psi = (\partial_\mu - \Gamma_\mu) \psi,$$

in tetrad- and coordinate-basis, respectively.

It is useful to remember that the absolute differential,  $DV_a \equiv D_c V_a ds^c$ , of a vector  $V_a$  is the principal linear part of the vector increment with respect to its change in the course of a parallel transport along the same infinitesimal path. Therefore, the parallel transport just means that  $DV_a = 0$ ; the vector does not change. The absolute differential of a Dirac field is needed for exactly the same reason. A stable spinor object does not change and it *is* parallel-transported along its world line. (As the matter of fact, the notion of parallel transport requires only one particular curve and a connection on this curve. The full congruence of curves in the vicinity of the path of parallel transport is not needed.) The tetrad representation most adequately reflects the local nature of vector fields in relativistic field theory. The law of the Lorentz transformation and the definition of parallel transport for spinors are far less obvious, because their components are not directly connected with the vectors of the coordinate axes. Therefore, spinors should always be treated as coordinate scalars. All Dirac matrices are treated as pure number constructs, which are the tools for certain substitutes of spinor components.

Although I called  $A_a(x)$  and  $\aleph_a(x)$  the fields, it seems more physical to give these designations to  $e(\rho_0)_{\alpha\beta} A_a = \delta_{\alpha\beta} e A_a$  and  $g(\rho_3)_{\alpha\beta} \aleph_a$ , the tetrad components of the spin connections. This is in line with the fact that the fields are always measured by their action on material bodies and that this action is always detected through the kinematics of the material bodies' motion. From this perspective, one may conclude that the physical meaning are not even the fields  $\delta_{\alpha\beta} e A_a$  and  $g(\rho_3)_{\alpha\beta} \aleph_a$  but rather the matrix elements of the full spin connection  $(\Gamma^a)_{\alpha\beta}$  between various states of the spinor field.

## B. Parallel transport of bilinear forms.

A summary of the computation of covariant derivatives for the sixteen basic bilinear spinor forms is as follows. If

$\mathcal{J}_a = \psi^+ \rho_3 \alpha_a \psi = \bar{\psi} \gamma^5 \gamma_a \psi$  is the axial current,  $\mathcal{S} = \bar{\psi} \psi = \psi^+ \rho_1 \psi$  and  $\mathcal{P} = \bar{\psi} \gamma^5 \psi = \psi^+ \rho_2 \psi$  are the two Lorentz scalars, and  $M_{ab}$  is a tensor with components  $M_{0i} = -i\psi^+ \rho_2 \sigma_i \psi = \bar{\psi} \gamma^0 \gamma^i \psi$  and  $M_{ik} = -i\epsilon_{ikm} \psi^+ \rho_1 \sigma_m \psi = \bar{\psi} \gamma^i \gamma^k \psi$ , then

$$\begin{aligned} D_\mu j^\nu &= \nabla_\mu j^\nu, & D_\mu \mathcal{J}^\nu &= \nabla_\mu \mathcal{J}^\nu, \\ D_\mu M^{\lambda\nu} &= \nabla_\mu M^{\lambda\nu} + g \aleph_\mu \epsilon^{\lambda\nu\rho\sigma} M_{\rho\sigma}, \\ D_\mu \mathcal{S} &= \nabla_\mu \mathcal{S} + 2g \aleph_\mu \mathcal{P}, & D_\mu \mathcal{P} &= \nabla_\mu \mathcal{P} - 2g \aleph_\mu \mathcal{S}. \end{aligned} \quad (2.3)$$

These equations are originally derived in the tetrad representation (when only spinors are differentiated,  $D\psi = D_a \psi ds^a$ ) and only after that are translated into coordinate form. The first equation duplicates the input for Eq. (2.1). The parallel transport of the vector currents that are built with the aid of diagonal Dirac matrices  $\sigma_i$  and  $\rho_3$  is not affected by the axial field. In the Lorentz scalars and the tensor, the left- and right- spinors are mixed by either  $\rho_1$  or  $\rho_2$  which makes their covariant derivative dependent of  $\aleph_\mu = e_\mu^a \rho_3 \aleph_a$ .

To derive equations similar to (2.3) for any other form  $O$ , defined by an operator  $\mathcal{O}$ , one has to follow a simple rule,

$$DO(x) \equiv D[\psi^+ \mathcal{O} \psi] = \psi^+ (\mathcal{O} \overrightarrow{D}_a + \overleftarrow{D}_a^+ \mathcal{O}) \psi ds^a. \quad (2.4)$$

Accordingly, the parallel transport of  $O(x)$  means that  $DO(x) = 0$ .

The vector and axial currents are the sum and the difference of the light-like left- and the right- currents of the Weyl spinors,  $j_a = j_a^L + j_a^R$  and  $\mathcal{J}_a = j_a^L - j_a^R$ . There also exists a well-known set of algebraic identities between various invariants of Dirac and Weyl spinors. It is straightforward to check that

$$\begin{aligned} \mathcal{J}_a j^a &= 0, & (\mathcal{S} \pm i\mathcal{P})^2 &= (\vec{L} \pm i\vec{K})^2, \\ j_a j^a &= -\mathcal{J}_a \mathcal{J}^a = 2j_a^L j^{Ra} = \mathcal{S}^2 + \mathcal{P}^2 > 0, \end{aligned} \quad (2.5)$$

where  $K_i = M_{0i}$ , and  $2L_i = \epsilon_{ikm} M_{km}$ . These formulae relate the tensor densities, and they are identically satisfied at every point.

## C. The Dirac equation revisited.

As was discussed in detail in paper [1], there is nothing at our disposal that could have been used to create an equation for the spinor field, except for the covariant derivative of a spinor field. In a tetrad basis it was found to be  $D\psi = D_a \psi ds^a = (\partial_a \psi - \Gamma_a \psi) ds^a$ , with the connection  $\Gamma_a$  given by equation (2.2). The structure of this spin connection is two-fold; its spin indices are being used to parameterize rotations of the local tetrad basis by means of the Pauli matrices, and its Lorentz index indicates the direction of parallel transport. The first step is to parameterize the Lorentz index  $a$  by a spinor and thus convert this derivative entirely into a spinor representation. This

conversion of two constituents of the Dirac spinor is carried out by means of the matrix  $\alpha_a = (1, \rho_3 \sigma_i)$ ; the left- and right- spinors are Lorentz transformed *differently*.

Taking the linear relation,  $u_\mu P^\mu = m$ , as a classical prototype for the equation of motion, we can write the following version of the Dirac equation (and its conjugate),

$$\begin{aligned} \alpha^a (\partial_a \psi - \Gamma_a \psi) + im \rho_1 \psi &= 0, \\ (\partial_a \psi^\dagger - \psi^\dagger \Gamma_a^\dagger) \alpha^a - im \psi^\dagger \rho_1 &= 0, \end{aligned} \quad (2.6)$$

with the spin connection (2.2), which includes an additional vector field  $\aleph_a(x)$  that acts differently on different spinor components. The differential operator of the Dirac equation (2.6) is Hermitian (symmetric), which is confirmed by the conservation of vector (probability) current  $j^\mu$ , but this does not necessarily mean that it is a self-adjoint operator. In fact, the axial potential can be so singular that, in general, it is not self-adjoint unless a special set of requirements is imposed on the axial field.

#### D. Conservation laws and equations of motion for the spin connections.

The equations of motion (2.6) allow one to derive a number of identities. One of them,

$$\nabla_\mu j^\mu = \nabla_\mu [\psi^\dagger \alpha^\mu \psi] = \frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} \psi^\dagger \alpha^\mu \psi] = 0, \quad (2.7)$$

clearly indicates the conservation of the time-like probability current and thus provides a definition of a scalar product in the space of Dirac spinor fields as an integral over the three-dimensional space-like surface. The second identity indicates that the axial current cannot be conserved,

$$\nabla_\mu \mathcal{J}^\mu = \nabla_\mu [\psi^\dagger \rho_3 \alpha^\mu \psi] = 2m \psi^\dagger \rho_2 \psi = 2m \mathcal{P}, \quad (2.8)$$

in full compliance with its space-like nature. Indeed, the conserved current can only be time-like. The pseudoscalar density is a measure of a dynamic interplay between the left- and right- components of the Dirac spinor (deviation from a perfect parity-even configuration encoded in the conventional Dirac equation [9, 10]).

Next, we introduce a standard energy-momentum tensor, which is conceived as a flux of the momentum,

$$T_\mu^\sigma = i \psi^\dagger \alpha^\sigma \overrightarrow{D}_\mu \psi. \quad (2.9)$$

Indeed,  $\alpha^\sigma$  is the quantum mechanical operator of the velocity and  $D_\mu$  is a prototype of the kinetic momentum  $P_\mu = m u_\mu$ . Its operator is supposed to be Hermitian *and* self-adjoint. Hence, it may seem reasonable to take  $T_\mu^\sigma$ , in advance, in a manifestly *real* form,  $2T_\mu^\sigma = i \psi^\dagger [\alpha^\sigma \overrightarrow{D}_\mu - \alpha^\sigma \overleftarrow{D}_\mu] \psi$ . However, it is not safe to have an operator that simultaneously acts in two adjoint spaces without confidence that these two spaces coincide.

One needs to know beforehand that the operator  $D_\mu$  is self-adjoint, which is not obvious because, in general, the axial potential in  $D_\mu$  is singular. The condition (4.8) for the self-adjointness of  $T_\mu^\sigma$  will be derived in Sec. IV, and will render the *ad hoc* symmetrization pointless.

It is straightforward to show (e.g., using the technique of Ref. [2]) that, by virtue of the equations of motion, the following identity holds

$$\begin{aligned} \nabla_\sigma T_\mu^\sigma &\equiv \frac{1}{\sqrt{-g}} \partial_\sigma [\sqrt{-g} T_\mu^\sigma] - \Gamma_{\mu\nu}^\sigma T_\sigma^\nu \\ &= i \psi^\dagger [D_\sigma D_\mu - D_\mu D_\sigma] \psi - 2mg \aleph_\mu \mathcal{P}. \end{aligned} \quad (2.10)$$

Here, the commutator of the covariant derivatives (the curvature tensor) has the following tetrad representation,

$$[D_\sigma, D_\mu] = e_\sigma^a [\partial_b \Gamma_a - \partial_a \Gamma_b + \Gamma_a \Gamma_b - \Gamma_b \Gamma_a + C_{ab}^c \Gamma_c] e_\mu^b,$$

where the structure constants  $C_{ab}^c = \omega_{ab}^c - \omega_{ba}^c$  can be related to some Lie group associated with parallel transport. An explicit computation shows that

$$\begin{aligned} \nabla_\sigma T_\mu^\sigma &= -e F_{\sigma\mu} j^\sigma - g \mathcal{U}_{\sigma\mu} \mathcal{J}^\sigma \\ &+ \frac{i}{4} \psi^\dagger \alpha_\sigma R_{\mu;cd}^\sigma \rho_1 \alpha^c \rho_1 \alpha^d \psi - 2gm \aleph_\mu \mathcal{P}, \end{aligned} \quad (2.11)$$

where the commutator of the covariant derivatives is expressed in terms of two gradient invariant tensors (the field strengths),  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ , and  $\mathcal{U}_{\mu\nu} = \partial_\mu \aleph_\nu - \partial_\nu \aleph_\mu$ , and the Riemann curvature tensor,

$$R_{ab;cd} = \partial_b \omega_{cda} - \partial_a \omega_{cdb} + \omega_{fca} \omega_{db}^f - \omega_{fcb} \omega_{da}^f + C_{ab}^f \omega_{fcd}.$$

It is straightforward to show that the third term in Eq. (2.11) can be transformed in the following way,

$$\psi^\dagger \alpha^a R_{\mu;cd}^\sigma \rho_1 \alpha^c \rho_1 \alpha^d \psi = 2R_{\mu\sigma} j^\sigma,$$

where  $R_{ad} = g^{bc} R_{ab;cd}$  is the Ricci curvature tensor [25].

The next step is to convert Eq. (2.11) into the divergence of one common energy momentum tensor for all fields in the system. Hence, we need the equations of motions for the fields  $A_\mu$  and  $\aleph_\mu$ . The definition of the field tensors immediately yields the first (without the sources) couple of the Maxwell equations for the field strengths  $F_{\mu\nu}$  and  $\mathcal{U}_{\mu\nu}$ ,

$$\nabla_\lambda \epsilon^{\sigma\lambda\mu\nu} F_{\mu\nu} = 0, \quad \nabla_\lambda \epsilon^{\sigma\lambda\mu\nu} \mathcal{U}_{\mu\nu} = 0.$$

The equations that interconnect fields and currents were motivated by the actual hydrogen spectra in the first paper. The Lorentz invariant form of the Coulomb law for the field  $A_\mu$  is

$$\nabla_\sigma F^{\sigma\mu} = e j^\mu = e [\psi^\dagger \alpha^\mu \psi], \quad (2.12)$$

so that  $F_{\mu\nu}$  is the massless gradient-invariant Maxwell field which has the *probability current* as its source. For the axial field  $\aleph_\mu$  a plausible choice is a massive neutral vector field,

$$\nabla_\sigma \mathcal{U}^{\sigma\mu} + M^2 \aleph^\mu = g \mathcal{J}^\mu, \quad M^2 \nabla_\mu \aleph^\mu = 2gm \mathcal{P}, \quad (2.13)$$

where the second equation is the covariant derivative of the first one.

The Lorentz forces in Eq. (2.10), with the field tensors  $F_{\mu\nu}$  and  $\mathcal{U}_{\mu\nu}$  in a familiar role, prompt the same equations of motion, because these equations allow one to present the Lorentz force as the divergence of the energy-momentum tensor. We have

$$ej^\sigma F_{\sigma\mu} = \nabla_\lambda [F^{\lambda\nu} F_{\nu\mu} + \frac{1}{4} \delta_\mu^\lambda F^{\rho\nu} F_{\rho\nu}] = \nabla_\lambda \Theta^\lambda{}_\mu. \quad (2.14)$$

Using (2.13) and (2.8) one can transform the Lorentz force of the axial field in (2.10) into the divergence of its energy-momentum tensor,

$$g\mathcal{J}^\sigma \mathcal{U}_{\sigma\mu} + g\aleph_\mu (\nabla_\sigma \mathcal{J}^\sigma) = \nabla_\lambda (\theta^\lambda{}_\mu + t^\lambda{}_\mu), \quad (2.15)$$

where

$$\begin{aligned} \theta^\lambda{}_\mu &= \mathcal{U}^{\lambda\nu} \mathcal{U}_{\nu\mu} + \frac{\delta_\mu^\lambda}{4} \mathcal{U}^{\rho\nu} \mathcal{U}_{\rho\nu}, \\ t^\lambda{}_\mu &= M^2 \left( \aleph^\lambda \aleph_\mu - \frac{\delta_\mu^\lambda}{2} \aleph^\rho \aleph_\rho \right). \end{aligned} \quad (2.16)$$

In both (2.12) and in (2.13) we followed the standard convention, which defines the charge as the divergence of its electric field, so that the positive charge corresponds to the positive flux of the electric field outside a surrounding surface. With this convention, the energy components  $\Theta^{00}$ ,  $\theta^{00}$  and  $t^{00}$  of tensors (2.14) and (2.16) come out *positive* exclusively because the coupling constants in the electron spin connection (2.2) were chosen *negative*.

For the reasons that will become clear later, the energy-momentum tensor of the axial field is split into parts with and without derivatives. The first part has the form of the Maxwell tensor. The second one *will acquire* derivatives *after* the condition of compactness (3.0) is imposed on the Dirac field. Putting Eqs. (2.10)-(2.16) together, one finds that

$$\nabla_\lambda (T^\lambda{}_\mu + \Theta^\lambda{}_\mu + \theta^\lambda{}_\mu + t^\lambda{}_\mu) = \frac{i}{2} R_{\mu\sigma} j^\sigma = 0. \quad (2.17)$$

The total energy-momentum of three interacting fields,  $\psi$ ,  $A_\mu$ , and  $\aleph_\mu$  is conserved, which is an additional indication that the system of equation of motion is self-consistent (provided this system has a time-like Killing vector field and the operator  $D_a$  is self-adjoint.) The energy-momentum tensors of both vector fields have *positive* energy densities, while the energy of the Dirac field can have both signs. The second equation in (2.17), namely,  $R_{\mu\sigma} = 0$ , will be derived in Sec. IV.

Formally, the equations of motion follow from the Lagrangian,  $\mathcal{L} = \mathcal{L}_D + \mathcal{L}_A + \mathcal{L}_\aleph$ , where the Lagrangians of the three fields,  $\psi$ ,  $A_a$  and  $\aleph_a$  are,

$$\mathcal{L}_D = i\psi^\dagger \alpha^\mu D_\mu \psi - m\psi^\dagger \rho_1 \psi$$

$$\mathcal{L}_A = -\frac{1}{4} F^{\rho\nu} F_{\rho\nu}, \quad \mathcal{L}_\aleph = -\frac{1}{4} \mathcal{U}^{\rho\nu} \mathcal{U}_{\rho\nu} - \frac{M^2}{2} \aleph^\rho \aleph_\rho,$$

and all fields at all space-time points are defined exclusively with respect to a local tetrad basis.

### III. COMPACT OBJECTS FROM DIRAC SPINORS

So far, we have dealt only with the definition of parallel transport, equations of motion and the most primitive conservation laws. Almost nothing has been said about the physical objects that are governed by these equations. Special relativity is built on two premises, the light-like propagation of all fields that carry a signal and the existence of inertial frames. The first one is readily implemented by a spinor representation of the Lorentz group – the two-component Weyl spinors just map the light cone. The second one is more difficult to implement because any inertial observer should have its own proper time and the solutions of the relativistic wave equations are not easily localized to the extent so that they can serve as the observers (rods and clocks) of special relativity. The electron with a given momentum is just a plane wave! At the same time, all data point to the fact that all truly localized interactions are due to spinor fields. Therefore, the major problem is to constructively identify the finite-sized spinor objects and a free space between them. Without clarity at this point the entire concept of the space-time continuum is vague and the idea of motion has no firm physical footing.

The spinor field that has the axial vector field in its spin connection seems to be more flexible than the original Dirac electron and more suitable to approach this old problem. To give an idea of how it might work, let us resort to the classical limit of a relativistic point-like particle with the “built in” polarization  $\vec{\zeta}$  [11]. If  $\vec{\zeta}$  of the rest frame is considered as a 4-vector, which is orthogonal to the 4-velocity,  $\zeta^\mu u_\mu = 0$ , then both vectors are readily converted into the vector fields *by means of* their parallel transport,

$$Du^\mu = (\partial_\nu u^\mu + \Gamma_{\lambda\nu}^\mu u^\lambda) dx^\nu = 0,$$

$$D\zeta^\mu = (\partial_\nu \zeta^\mu + \Gamma_{\lambda\nu}^\mu \zeta^\lambda) dx^\nu = 0.$$

We can continue by taking  $dx^\nu = u^\nu d\tau$  and end up with the equations for the geodesic trajectory of a point-like particle and the parallel transport of its polarization along this trajectory. The vector  $\vec{\zeta}$  is a classical prototype for the *internal polarization* of the Dirac spinor field which is represented by various bilinear forms, like  $\mathcal{S}$ ,  $\mathcal{P}$ , etc. In the context of the Dirac equation, the vector  $\vec{\zeta}$  is associated with the components of the tensor  $M_{\mu\nu}$  where the left- and right- spinors are mixed by the matrices  $\rho_1$  and  $\rho_2$ . As long as we wish to treat the electron as a compact object with a frozen-in polarization (including all its quantum numbers) we have to *require* that these attributes are parallel transported,

$$DS = 0, \quad D\mathcal{P} = 0, \quad \dots, \quad (3.0)$$

according to the definition (2.4). The goal of this *ansatz* is to put a compact object built entirely from spinor fields

(and their spin connections) into an inertial frame that accompanies it. The spin connection of the Dirac field remains non-integrable,  $D\psi = (\partial_a - \Gamma_a)\psi ds^a \neq 0$ . The ansatz (3.0) serves as an additional restriction on the possible field  $\aleph_a$  in close proximity of a stable spinor object.

### A. Gordon's decomposition

A list of attributes of a compact object that should be subjected to the *ansatz* (3.0) can be identified by means of the so-called Gordon's decomposition. In many cases,

it draws a clear cut distinction between the convection and polarization currents. Using spin connection (2.2) and the algebraic relations,

$$\alpha^c \rho_1 \alpha^d + \alpha^d \rho_1 \alpha^c = 2g^{cd} \rho_1,$$

and

$$\alpha^c \rho_1 \alpha^d - \alpha^d \rho_1 \alpha^c = 2\rho_1 \Sigma^{cd},$$

where  $\Sigma^{cd} = (1/2)[\gamma^c, \gamma^d]$ , it is straightforward to present the probability current,  $j_a = \psi^\dagger \alpha^a \psi$ , as follows,

$$j_a = \frac{i\hbar}{2mc} \left\{ \psi^\dagger \rho_1 \overset{\leftrightarrow}{\partial}_a \psi - 2ieA_a \psi^\dagger \rho_1 \psi + \partial_b(\psi^\dagger \rho_1 \Sigma^{ab} \psi) - 2ig\aleph_b \psi^\dagger \rho_1 \Sigma^{ab} \rho_3 \psi + \frac{1}{4}\omega_{cdb} \psi^\dagger \rho_1 [\Sigma^{ab}, \Sigma^{cd}] \psi \right\}. \quad (3.1)$$

By virtue of the identity,  $[\Sigma^{ab}, \Sigma^{cd}] = 2[g^{ad}\Sigma^{bc} + g^{bc}\Sigma^{ad} - g^{ac}\Sigma^{bd} - g^{bd}\Sigma^{ac}]$ , we have  $(1/4)\omega_{cdb}[\Sigma^{ab}, \Sigma^{cd}] = \Sigma^{ac}\omega_{bcb} - \Sigma^{bc}\omega_{acb}$ , and the probability current becomes

$$j_a = \frac{i\hbar}{2mc} \left\{ [\psi^\dagger \rho_1 \partial_a \psi - (\partial_a \psi^\dagger) \rho_1 \psi - 2ieA_a \psi^\dagger \rho_1 \psi] + D_b(\psi^\dagger \rho_1 \Sigma^{ab} \psi) \right\}, \quad (3.2)$$

where  $D_b[\psi^\dagger \rho_1 \Sigma^{ab} \psi] = \nabla_b(\psi^\dagger \rho_1 \Sigma^{ab} \psi) - g\aleph_b \epsilon^{abcd} \psi^\dagger \rho_1 \Sigma_{cd} \psi$  is the absolute derivative of the polarization tensor. A similar representation is possible for the axial current,  $\mathcal{J}_a = \bar{\psi} \gamma^a \gamma^5 \psi$ ,

$$\mathcal{J}_a = \frac{-\hbar}{2mc} \left\{ [\psi^\dagger \rho_2 \Sigma^{ab} (\vec{\partial}_b - \Omega_b) \psi - \psi^\dagger (\vec{\partial}_b - \Omega_b) \rho_2 \Sigma^{ab} \psi + 2ieA_b(\psi^\dagger \rho_2 \Sigma^{ab} \psi)] + D_a(\psi^\dagger \rho_2 \psi) \right\}. \quad (3.3)$$

The original “diagonal” representation of the probability current  $j_a$  (that does not mix left- and right- spinors) is traded for a sum (3.2) of the “off-diagonal” terms (where these spinors are mixed). The first three terms (in brackets) is the convection current of the Schrödinger equation for a structureless particle, the last term is due to the internal polarization that is transported with the particle. In the axial current, the convection “drags” the density of internal polarization, while the pseudoscalar density  $\mathcal{P}$  is parallel transported with the particle, and it cannot change unless this particle is subjected to an external field or decays. Quite remarkably, the convective derivative includes only the Maxwell field, which acts on a Dirac particle as a whole, while the parallel transported polarization terms are affected only by the axial field, which differentiates between the right and left spinor components.

### B. Axial field of a compact Dirac particle

The ansatz (3.0) serves as an additional condition on the spin connection  $\rho_3 \aleph_a(x)$  of the Dirac field, which makes it compatible with the existence of a freely moving stable spinor particle. With an *a priori* form (2.2)

of the spin connection, it can be cast into the following coordinate form,

$$\begin{aligned} D_\mu \mathcal{S} &= \nabla_\mu \mathcal{S} + 2g\aleph_\mu \mathcal{P} = 0, \\ D_\mu \mathcal{P} &= \nabla_\mu \mathcal{P} - 2g\aleph_\mu \mathcal{S} = 0. \end{aligned} \quad (3.4)$$

(If there were no axial field  $\aleph_\mu$ , then we would have only one condition,  $\nabla_\mu M^{\lambda\nu} = 0$  (cf. Eq. (3.1)). As one can anticipate, the electric polarization current in free space is zero,  $j_{\text{polariz}}^\nu = \nabla_\lambda M^{\lambda\nu} = 0$ .)

Acting on Eqs. (3.4) by  $\nabla_\mu$ , and excluding the first derivatives we obtain the equations,

$$\begin{aligned} \square \mathcal{S} + 4g^2 \aleph^2 \mathcal{S} &= -(4g^2 m/M^2) \mathcal{P}^2, \\ \square \mathcal{P} + 4g^2 \aleph^2 \mathcal{P} &= (4g^2 m/M^2) \mathcal{S} \mathcal{P}. \end{aligned} \quad (3.5)$$

Eqs. (3.5) do not depend on the Maxwell field  $A_\mu$  and are the relativistic wave equations. This means that the propagation of static polarization of an isolated compact object can be reduced to a Lorentz transformation of the proper fields, which are completely defined in its rest frame. The densities  $\mathcal{S}, \mathcal{P}, \dots$  become the *effective fields* solely because we wish a compact spinor object to have a co-moving frame! Then *for any short-scale dynamics that might take place in this frame, the translational invariance must be explicitly broken*. From this viewpoint, the

function  $\aleph^2 = \aleph_0^2 - \vec{\aleph}^2$  in Eqs. (3.5) stands for a scalar potential and it determines if the “center of a particle” is attractive or repulsive for the fields of polarization and, eventually, for the other particles. The densities  $\mathcal{S}$  and  $\mathcal{P}$ , are the simplest possible ones. Any of them can serve as a mass term and provide the primary binding of two Weyl spinors.

As one can anticipate, the ansatz (3.0) indeed affects the possible form of the connection  $\aleph^a$  (and only of  $\aleph^a$ ). Instead of being an arbitrary external vector field it becomes a functional  $\aleph_a\{\mathcal{S}, \mathcal{P}\}$  of spinor forms: from Eqs. (3.4) it immediately follows that

$$2g\aleph_\mu\{\mathcal{S}, \mathcal{P}\} = \partial_\mu[\arctan(\mathcal{P}/\mathcal{S})] = -\partial_\mu\Upsilon\{\mathcal{S}, \mathcal{P}\}. \quad (3.6)$$

Alternatively, one can use Eqs. (3.4) to show that  $\mathcal{S}\partial_\mu\mathcal{S} + \mathcal{P}\partial_\mu\mathcal{P} = 0$ . Hence, the scalar function  $\mathcal{R}^2 = \mathcal{S}^2 + \mathcal{P}^2$ , is the first integral of these two equations (for a toy model of the localized solution with this property see Sec. V). This had to be expected because the current  $j^a$  is time-like and its conservation means that  $D_a j^a = 0$ . Since  $\mathcal{S}$  and  $\mathcal{P}$  are real functions, we may look for a solution of the form

$$\mathcal{S} = \mathcal{R} \cos \Upsilon, \quad \mathcal{P} = -\mathcal{R} \sin \Upsilon,$$

which yields the same equation as Eq. (3.6),

$$2g\aleph_\mu = -\partial_\mu\Upsilon. \quad (3.7)$$

Since  $\mathcal{R}^2 = j^a j_a$  is the squared probability current, the positive  $\mathcal{R}$  is a natural measure for *localized* spinor matter. *The potential function  $\Upsilon(x)$  remains a distinct characteristics of the Dirac field even in that part of space where  $\mathcal{R} \rightarrow 0$ .* For perfectly stable spinor matter, the axial field  $\aleph_\mu$  becomes very simple; it is completely defined by a single scalar function[26]. Its curvature tensor vanishes,  $\mathcal{U}^{\sigma\mu} = 0$ . Furthermore, if the conditions (3.0) are exact, then the Eqs. (2.16) become [27]

$$\theta^\lambda_\mu = 0, \quad t_{\lambda\mu} = \frac{M^2}{4g^2} \left( \partial_\lambda \Upsilon \partial_\mu \Upsilon - \frac{g_{\lambda\mu}}{2} \partial_\rho \Upsilon \partial^\rho \Upsilon \right). \quad (3.8)$$

This is the energy-momentum tensor of the massless scalar field  $\Upsilon(x)$  with a positively defined Hamiltonian. Thus, the full field  $\aleph$  has lost its transverse part but it still remains a dynamical object. The equation for this field can be found by computing the covariant divergence of Eq. (3.7) and expressing the divergence of the axial field via the pseudoscalar density,  $M^2 \nabla_\mu \aleph^\mu = 2gm\mathcal{P} = -2gm\mathcal{R} \sin \Upsilon$ . We find that

$$\square \Upsilon(x) - \frac{4g^2 m}{M^2} \mathcal{R}(x) \sin \Upsilon(x) = 0. \quad (3.9)$$

Within a domain where  $\mathcal{R}(x)$  is a constant, this is a well-known sine-Gordon equation which has the soliton solutions. This equation is classical, *the Planck constant is gone*; therefore, the field  $\Upsilon(x)$  does not vanish in the classical limit. In the case of a variable source, the variations

of  $\Upsilon(x)$  propagate according to the d’Alembert equation. It is not clear if this equation has static solutions at all. However, if such solutions do exist, then of particular interest is the case of spherical symmetry. If the exterior of the domain  $r < r_{max}$  is empty space where  $\mathcal{R}^2 = 0$ , then the function  $\Upsilon$  is a solution to the external problem of the Laplace equation[28]. Therefore, at  $r > r_{max}$  we have

$$\Upsilon(r) = -\frac{1}{r} \frac{4g^2 m}{M^2} \int_0^{r_{max}} \mathcal{P}(r) r^2 dr. \quad (3.10)$$

Now, the most convincing argument in favor of a strong localization comes from the Dirac equation itself, where the potential  $-\partial_r \Upsilon \sim -1/r^2$  enters the radial component  $ig\rho_3 \aleph_r$  of the spin connection and  $\aleph_a(x)$  is the remaining (less singular) part of the axial field which is not regulated by the ansatz and has a non-vanishing field tensor  $\mathcal{U}_{ab}$ ,

$$\alpha^a \left\{ \partial_a + ieA_a(x) + ig\rho_3 \aleph_a(x) + \frac{i}{2} \rho_3 \frac{\partial \Upsilon(x)}{\partial x^a} - \Omega_a(x) \right\} \psi + im\rho_1 \psi = 0. \quad (3.11)$$

The most singular potential  $\partial_a \Upsilon$  brings in the term  $\propto 1/r^4$  into an equivalent Schrödinger equation signaling an advent of the “falling onto the centre” phenomenon (and there is no threshold condition like  $Z > 137$  in the Coulomb field). An obvious conjecture is that this supercritical binding is likely to initiate a (rich) spectrum of (quasi)bound states where the Dirac field has a *negative energy*.

We may generalize this observation by realizing that, by a formal integration of Eq. (3.9) and expressing  $\Upsilon(x)$  in the Dirac equation (3.11) as an integral of  $\mathcal{P}$ , we convert the latter into a non-linear integral-differential equation for the spinor field  $\psi$ . In this equation, the mass parameter  $m$  will no longer be an arbitrary number.

One of the solutions of the coupled equations (3.9) and (3.11) is obvious. This is the solution with  $m = 0$  (formally) or, equivalently, with  $\mathcal{R}(x) = 0$  (physically). The probability current for this spinor mode is light-like. Such a mode can have only one, left- or right- spinor component, and it completely detaches from the field  $\Upsilon(x)$  (except for the obvious effect of the metric that will be derived in the next section).

#### IV. AXIAL FORCES AND GRAVITY.

The requirement (3.0) that the internal polarization densities like  $\mathcal{S}$  and  $\mathcal{P}$  must be frozen into a stable Dirac particle has limited the form of the axial field in the spin connection to a gradient of the potential function  $\Upsilon$ . This step has also led to a non-linear system of equations (3.9)-(3.11) that can yield a hierarchy of scales. Thus, this system can have localized solutions which represent compact clusters of the spinor field.

### A. Localization of energy and the force of inertia

Any compact object must also localize its energy-momentum along the world line of a particle. Therefore we are forced to augment the ansatz (3.0) by a new element,

$$D_\sigma T_\mu^\sigma = 0. \quad (4.0)$$

This is an expression of the fact that the kinetic 4-momentum is parallel-transported with the particle and does not change when it is moving in free space. By the definition, we have

$$D_\sigma T_\mu^\sigma \equiv i(\psi^+ \overleftarrow{D}_\sigma^\dagger \alpha^\sigma \overrightarrow{D}_\mu \psi + \psi^+ \alpha^\sigma \overrightarrow{D}_\mu \overrightarrow{D}_\sigma^\dagger \psi) = 0, \quad (4.1)$$

which yields the equation,

$$\frac{1}{\sqrt{-g}} \partial_\sigma (\sqrt{-g} T_\mu^\sigma) = i\psi^+ \alpha^\sigma [D_\mu D_\sigma - D_\sigma D_\mu] \psi. \quad (4.2)$$

It has to be compared with the identity (2.10) that follows from the equations of motion,

$$\begin{aligned} \frac{1}{\sqrt{-g}} \partial_\sigma [\sqrt{-g} T_\mu^\sigma] &= \Gamma_{\mu\nu}^\sigma T_\sigma^\nu \\ &+ i\psi^+ [D_\sigma D_\mu - D_\mu D_\sigma] \psi - 2mg\aleph_\mu \mathcal{P}. \end{aligned} \quad (4.3)$$

These two equations coincide if the force of inertia and the external force from axial field  $\aleph$  are equal, i.e.,

$$\Gamma_{\mu\nu}^\sigma T_\sigma^\nu = 2mg\aleph_\mu \mathcal{P} = -m\mathcal{P} \partial_\mu \Upsilon. \quad (4.4)$$

While the first part (3.0) of the ansatz simplified the spin connection  $\aleph_\mu$  to the gradient of a scalar function, the second part (4.0) specifies the affine connection. Indeed, the Ricci rotation coefficients  $\omega_{cda}$  that enter the spin connection  $\Omega_a$  are related to the Christoffel symbols  $\Gamma_{\mu\nu}^\sigma$  through the covariance of the tetrad vectors,  $D_b e_\mu^a = 0$ . The simplest form of Eq. (4.4) is  $T^{00} \partial_i g_{00} = 2mc^2 \mathcal{P} \partial_i \Upsilon$ , which immediately leads to

$$g_{00} = 1 + 2 \frac{m\mathcal{P}}{T_{00}} \Upsilon \rightarrow 1 + \frac{2\Upsilon_{Newton}}{c^2}.$$

Thus, we have *derived* the key formula, which is traditionally considered as a *phenomenological* basic of general relativity, and thus the Newtonian form (3.10) of the field  $\aleph$  at large distances is not a mere coincidence. The field  $\aleph_\mu$ , by all its properties, is indistinguishable from the gravitational field, which is also locally equivalent to the field of the inertia forces. Indeed, in a local co-moving (geodesic) reference frame, where the Christoffel connections  $\Gamma_{\mu\nu}^\sigma$  vanish, the potential  $\Upsilon$  must become constant (up to small second order corrections). Furthermore, the pseudoscalar density  $\mathcal{P}$  of a free Dirac plane wave is zero. The spinor field of a plane-wave electron is perfectly parity-even (see [9, 10]), and its “gravitational mass”  $\mathcal{P}$  is effectively switched off (it cannot be switched off globally as long as *two* bodies interact and accelerate).

Eq. (4.4) explicitly states that for a small body, which moves under the action of the axial field, it is possible to choose such a parameterization of the space-time coordinates (the metric tensor  $g_{\mu\nu}(x)$ ) that this body will move along the geodesic lines of *this* metric. In such a parameterization, the affine connections  $\Gamma_{\mu\nu}^\sigma$  take a role of the forces of inertia that locally compensate the physical axial forces. This substitution is impossible for the most general axial and spinor fields. The latter ones must form stable compact objects, which renders the axial-Newton’s gravitational field  $\Upsilon$  classical. Surprisingly enough, this predetermines a possible form of the metric tensor. The ansatz (3.0), (4.0) completely breaks up for unstable objects.

### B. Self-adjointness and the Einstein field equations

In the previous sections, keeping focus on the physical picture, we manipulated the operators of the Dirac equation and the energy-momentum tensor without confidence that these operators even exist. At the same time, the axial field  $\aleph$  was shown to be singular, which means that the spaces of spinor functions  $\psi$  and  $\psi^+$  can be different. Therefore, we have to demand that the above operators are self-adjoint.

The differential operator  $iD_\mu$  of the Dirac equation is Hermitian, which leads to the conservation of the probability current  $j^\mu$ . It is self-adjoint if

$$\begin{aligned} i \int \partial_\sigma (\sqrt{-g} \psi^+ [\alpha^\sigma \overrightarrow{D}_\mu + \overleftarrow{D}_\mu^\dagger \alpha^\sigma] \psi) d^3 \vec{x} dx^0 \\ = i \int \partial_\sigma \nabla_\mu (\sqrt{-g} \psi^+ \alpha^\sigma \psi) d^3 \vec{x} dx^0 = 0. \end{aligned} \quad (4.5)$$

This Lorentz invariant equation encodes two requirements. First, that the self-adjointness is defined with the scalar product as an integral over a space-like surface. Second, that this condition is the same for all space-like surfaces.

The Dirac equation (3.11) can have stable compact solutions with real energies only when the operator of energy-momentum is self-adjoint. Then the solutions of the Dirac equation and its adjoint belong to the same space (e.g., have the same spectra of energies). This condition has to be parallel-transported with the compact object that “owns” this spectrum, and the covariant form of this requirement is

$$\begin{aligned} D_\sigma (\psi^+ [\alpha^\sigma \overrightarrow{D}_\mu + \overleftarrow{D}_\mu^\dagger \alpha^\sigma] \psi) &= \psi^+ [\alpha^\sigma \overrightarrow{D}_\mu + \overleftarrow{D}_\mu^\dagger \alpha^\sigma] \overrightarrow{D}_\sigma \psi \\ &+ \psi^+ \overleftarrow{D}_\sigma^\dagger [\alpha^\sigma \overrightarrow{D}_\mu + \overleftarrow{D}_\mu^\dagger \alpha^\sigma] \psi = 0, \end{aligned} \quad (4.6)$$

which can be identically transformed into

$$\begin{aligned} \partial_\sigma (\sqrt{-g} [\psi^+ \alpha^\sigma \overrightarrow{D}_\mu \psi + \psi^+ \overleftarrow{D}_\mu^\dagger \alpha^\sigma \psi]) \\ + R_{\mu\sigma} \sqrt{-g} \psi^+ \alpha^\sigma \psi = 0, \end{aligned} \quad (4.7)$$

then integrated over the space-time domain, and compared with the condition (4.5). This leads to a conclusion

that the Dirac field can have a self-adjoint Hamiltonian only if it lives in a *free space* (as it is understood in general relativity). Since  $j^\mu \neq 0$ , we must have

$$R_{\lambda\sigma} = 0. \quad (4.8)$$

In general relativity theory, this is Einstein's field equation. The Dirac spinor field, taken as matter, not only leads to this equation, but allows one to derive the principle of inertia and establish the identity between the axial and gravitational fields.

Several remarks are in order.

- (i) The ansatz (3.0) and (4.0) together with equation (4.4) are the spinor equivalent of the Fermi-Walker transport of tensor fields [12] which are confined to the closest vicinity of the particle's world line.
- (ii) The origin of Eq. (4.8) in the context of this work clearly supports Einstein's ultimate opinion that the energy-momentum tensor does not represent the gravitating matter adequately. It also supports Einstein's later attempts to identify matter with singularities of the field equations (4.8) [5]. The major result was that, even considered separately, the field equations for the metric of free space have *singularities that are moving according to the laws of motion of classical particles*. The principle of motion along a geodesic line follows from the field equations and is not an independent principle.
- (iii) Equations (3.9), (3.11) and (4.8), (4.4) clearly support the image of particles as moving singularities of the *interacting Dirac and gravitational fields*. The gravitational force appears to be a spin connection of the Dirac field. In fact, this system guarantees that the theory will be protected from *mathematical singularities* which inevitably show up when the nonlinear Einstein's equations (4.8) are solved independently [5]. A striking agreement between the character of motion of singular domains of the Einstein and Dirac fields seems to be an indication that general relativity naturally requires matter in the form of the Dirac field. No other fields can provide the degree of localization, which is necessary for such a coincidence. The complementarity of these two approaches indeed solves the problem of motion as it was first posed by Einstein.

## V. LOCALIZATION OF THE DIRAC FIELD AND THE GRAVITATIONAL MASS.

In this section, the toy model of a compact Dirac particle will be worked out. This model captures some essential properties of the yet unknown exact solutions of the non-linear system of Eqs. (3.9), (3.11) and (4.8). The model employs a symmetric closed configuration of the Dirac field, which corresponds to the previously conjectured [1] possibility that the Dirac field is radially polarized in the internal geometry of a sphere. It is an eigenstate of the operator  $\sigma_3$  associated with the radial tetrad vector  $e_a^3$ , so that the Dirac spinor is parallel-transported along a sphere. One of the possible prototypes of such

a state is the fully occupied electron shell of a noble gas, which is so symmetric that none of the electrons can be assigned individual quantum numbers associated with the angular momentum. Another example is the linear oscillatory orbit of the Bohr-Sommerfeld model. The only topologically distinctive direction is the radial one, inward or outward.

Strictly speaking, considering this static model more seriously, one has to use the Schwarzschild solution as the uniquely defined (according to the Birkhoff theorem) metric background. According to Eq. (4.4), the gravitational radius of this metric is defined by the distribution of the pseudoscalar density  $\mathcal{P}$ , the peak localization of the Dirac field, rather than by its energy (inertial mass). Any attempt to incorporate these elements leads to the issue of the very existence of the meaningful (measurable) metric relations near the Planck scale and, possibly, even of the origin of the quantum-mechanical ensembles [14], which is beyond the scope of this paper.

Any extension of this model will also include modes with angular dependence and, perhaps, some relevant quantum numbers. These modes can have special names, e.g.  $\{R, G, B\}$ , but these names cannot be given to plane waves – they are meaningful only in the internal space of closed configurations. As it was pointed out in paper [1], in the presence of the axial field the angular momentum is not conserved even in a perfectly spherical geometry. This fact points to the possibility that in the course of the formation of a closed spinor configuration some of the polarization degrees of freedom undergo a transition from the “normal” world (of the directions to distant stars) into a compactified internal space. This internal space (invisible from the outside) is very likely to have a non-Abelian symmetry group and, possibly, a local gauge structure associated with it. Therefore, the well known non-Abelian gauge theories may have a natural realization within this scheme. The radial modes of these configurations can also have special names, e.g.,  $\{u, d, \dots\}$ . The study of the emerging non-Abelian structure, similar to the one which was pioneered in Ref. [15], is currently underway.

From Eqs. (4.11) and (4.14) of the paper [1], we have two systems of equations for two topologically distinct modes

Equations for the modes with outward polarization are

$$\begin{aligned} [E_\uparrow - \frac{Ze^2}{r} - g\aleph_r - i\partial_r]f_\uparrow &= mh_\uparrow, \\ [E_\uparrow - \frac{Ze^2}{r} - g\aleph_r + i\partial_r]h_\uparrow &= mf_\uparrow. \end{aligned} \quad (5.1)$$

where the functions  $f$  and  $h$  include the factor  $(-g)^{1/4} = r\sqrt{\sin\theta}$  from the Jacobian, so that the measure of the volume integration is  $drd\theta d\phi$  (at this instance only we include in these equations the attractive Coulomb potential of the point-like charge  $+Ze$ , as a reference). For inward modes we have the same system with the opposite

sign of  $\aleph_r$ . Introducing the new functions,

$$\sqrt{2}F = f + h \quad \text{and} \quad i\sqrt{2}G = f - h,$$

we obtain equations with real coefficients,

$$F'_\uparrow = [m + E_\uparrow - g\aleph_r]G_\uparrow, \quad G'_\uparrow = [m - E_\uparrow + g\aleph_r]F_\uparrow, \quad (5.2)$$

where  $F' = dF/dr$ . For both modes (and assuming real solutions) we have the identity

$$(F^2 + G^2)' = 4mFG. \quad (5.3)$$

(In our simple case, it can be traced back to  $\nabla_\mu \mathcal{J}^\mu = 2m\mathcal{P}$  of Eq. (2.8) and  $\mathcal{J}^2 = -j^2$  of Eq. (2.5)). This identity allows one to find the volume integral of pseudoscalar density by means of the relations,

$$\mathcal{P} = \mathcal{P}_\uparrow + \mathcal{P}_\downarrow, \quad \sqrt{-g} \mathcal{P}_{\uparrow\downarrow} = \mp 2F_{\uparrow\downarrow}G_{\uparrow\downarrow}. \quad (5.4)$$

Let us think of a localized distribution of the Dirac field as a source of a static potential  $\aleph_r^{(0)}(r)$ . Then, according to Eq. (3.10),

$$\begin{aligned} g\aleph_r^{(0)\dagger}(r) &= -\frac{1}{r^2} \frac{2mg^2}{M^2} \int_0^{r_{max}} \mathcal{P}_\uparrow(r)r^2 dr \\ &= \frac{1}{r^2} \frac{g^2}{M^2} \int_0^{r_{max}} (F_\uparrow^2 + G_\uparrow^2)' dr. \end{aligned}$$

Since the object is localized, there is no contribution from the upper limit

$$g\aleph_r^{(0)\dagger}(r) = -\frac{1}{r^2} \frac{g^2}{M^2} R_\uparrow(0) = -\frac{\mathcal{Q}_\uparrow}{r^2}, \quad (5.5)$$

where  $R(r)drd\Omega = r^2\mathcal{R}(r)drd\Omega$  is the physical probability density. (For a  $\downarrow$ -center we obviously have to change sign in the r.h.s.) If this probability is normalized to unity within a sphere of radius  $r_{max}$ , then  $R(0) \sim 1/r_{max}$  and, according to Eq. (5.12), this estimate can be very accurate. This provides a simple formula,

$$2m \int \mathcal{P}_\uparrow dV \approx R(0). \quad (5.6)$$

Physically, this quantity represents the gravitational mass in Eqs. (3.10), (4.3), and (4.4) clearly indicating that, at the microscopic level, the strength of the axial-Newton interaction is proportional to the *peak of localization* of the Dirac field at the gravitating centers.

The next step is to solve Eqs. (5.1) with these external sources. For the sake of definiteness, let us take (5.5) as an external field. Then Eqs. (5.1) and (5.2) with  $\uparrow$ -center are

$$\begin{aligned} [E_\uparrow + \frac{\mathcal{Q}_\uparrow}{r^2} - i\partial_r]f_\uparrow &= mh_\uparrow, \quad [E_\uparrow + \frac{\mathcal{Q}_\uparrow}{r^2} + i\partial_r]h_\uparrow = mf_\uparrow, \\ F'_\uparrow &= [m + E_\uparrow + \frac{\mathcal{Q}_\uparrow}{r^2}]G_\uparrow, \quad G'_\uparrow = [m - E_\uparrow - \frac{\mathcal{Q}_\uparrow}{r^2}]F_\uparrow, \end{aligned} \quad (5.7)$$

Their primary form (the upper line) indicates that we have the Dirac equation with a *repulsive* singular potential. The true degree of singularity is seen from the second order equation, i.e.,

$$f'' + \left[ \frac{2i\mathcal{Q}}{r^3} + \left( E - \frac{\mathcal{Q}}{r^2} \right)^2 - m^2 \right] f = 0, \quad (5.8)$$

(which is similar to the one studied in Ref. [16]).

Eqs. (5.7) depend on three dimensional parameters, which are the mass parameter  $m$ , the energy  $E$  and the intensity of the central charge  $\mathcal{Q}$  (with the dimension  $m^{-1}$ ). Looking for the simplest possible object, I shall retain only two of them (by taking  $E = m$  or  $E = -m$ ). This will render Eqs. (5.7) easily solvable. Indeed, for the most interesting case,  $E = -m < 0$ , we have

$$G'_\uparrow = \left[ 2m - \frac{\mathcal{Q}_\uparrow}{r^2} \right] F_\uparrow, \quad r^2 F'_\uparrow = \mathcal{Q}_\uparrow G_\uparrow. \quad (5.9)$$

Differentiating the second equation, using the first one, and changing the independent variable to  $y = 1/r$ , we obtain

$$\frac{d^2 F_\uparrow}{dy^2} + \left( \mathcal{Q}_\uparrow^2 - \frac{2m\mathcal{Q}_\uparrow}{y^2} \right) F_\uparrow = 0, \quad G_\uparrow = \frac{-1}{\mathcal{Q}_\uparrow} \frac{dF_\uparrow}{dy}. \quad (5.10)$$

The solution that vanishes when  $y \rightarrow 0$  ( $r \rightarrow \infty$ ) at all negative energies is

$$\begin{aligned} F_\uparrow(y) &= C_\nu y^{1/2} J_\nu(\mathcal{Q}_\uparrow y) = \frac{C_\nu}{r^{1/2}} J_\nu\left(\frac{\mathcal{Q}_\uparrow}{r}\right), \\ \nu^2 &= \frac{1}{4} + 2m\mathcal{Q}_\uparrow > 0. \end{aligned} \quad (5.11)$$

The normalization integral,  $\int [F^2 + G^2] dr$ , converges (at  $r \rightarrow \infty$ ) only when  $\nu > 1$ , and the normalization coefficient is

$$C_\nu = \left( \frac{16\nu(\nu^2 - 1)}{3(4\nu^2 - 1)} \right)^{1/2}.$$

The behavior of this solution as  $r \rightarrow 0$  is noteworthy:

$$F \approx C_\nu \sqrt{\frac{2}{\pi\mathcal{Q}_\uparrow}} \cos \frac{\mathcal{Q}_\uparrow}{r}, \quad G \approx C_\nu \sqrt{\frac{2}{\pi\mathcal{Q}_\uparrow}} \sin \frac{\mathcal{Q}_\uparrow}{r}, \quad (5.12)$$

so that the probability density  $F^2 + G^2$  remains surprisingly constant within the range of validity of the asymptotic formula for the Bessel functions. These normalizable bound states occupy the range of energies  $-\infty < E_\uparrow < -E_* = -3/(8\mathcal{Q}_\uparrow)$ , and their energy spectrum is *continuous*. In the energy interval  $E_* < E < 0$ , the solution still goes to zero as  $r \rightarrow \infty$ , but not sufficiently fast as to be a true bound state without an additional cutoff (which is easily provided, e.g., by an attractive Coulomb potential).

In exactly the same way, we consider the case  $E_\uparrow = m > 0$ , which leads to the equations

$$\frac{d^2 G_\uparrow}{dy^2} + \left( \mathcal{Q}_\uparrow^2 + \frac{2m\mathcal{Q}_\uparrow}{y^2} \right) G_\uparrow = 0, \quad F_\uparrow = \frac{1}{\mathcal{Q}_\uparrow} \frac{dG_\uparrow}{dy}. \quad (5.13)$$

The only solution to this equation which vanishes when  $y \rightarrow 0$  ( $r \rightarrow \infty$ ) is

$$G_{\uparrow}(y) = C_{\mu} y^{1/2} J_{\mu}(\mathcal{Q}_{\uparrow} y), \quad \mu^2 = \frac{1}{4} - 2m\mathcal{Q}_{\uparrow} > 0. \quad (5.14)$$

At  $r \rightarrow 0$  ( $y \rightarrow \infty$ ) this solution also infinitely oscillates, which indicates a “fall onto a centre”. Therefore, this is a solution confined around a seemingly repulsive core and is a quasi-bound state when  $0 < E_{\uparrow} < E_{\#} = 1/(8\mathcal{Q}_{\uparrow})$ . Since  $\mu^2 < 1$ , it is not normalizable as a true bound state. In the opposite case, when  $E_{\uparrow} > E_{\#}$ , we have  $\lambda^2 = -\mu^2 > 0$ . A general solution,

$$G_{\uparrow}(y) = C y^{1/2} [e^{i\beta} J_{i\lambda}(\mathcal{Q}_{\uparrow} y) + e^{-i\beta} J_{-i\lambda}(\mathcal{Q}_{\uparrow} y)], \quad (5.15)$$

is not localized, unless some sort of boundary condition is imposed. Any particular choice of these conditions will correspond to a special self-adjoint extension of the Dirac operator and a special physical input.

We may summarize the above analysis as follows. The  $\uparrow$ -Dirac mode in the field of the singular repulsive potential  $+Q_{\uparrow}/r^2$  has bound states of energies  $-3/8 < E_{\uparrow} \mathcal{Q}_{\uparrow} < -\infty$ . Furthermore, this field creates its own distribution of pseudoscalar density  $\mathcal{P}_{\uparrow}$  which tends to amplify the external potential and to push the field into stronger localization at *negative energies* (the upper bound  $E_{*}$  approaches zero when  $\mathcal{Q}_{\uparrow}$  increases).

Following the same procedure, we may solve the Eqs. (5.1) and (5.2) for the  $\uparrow$ -mode in the field of a  $\downarrow$ -center,

$$\begin{aligned} [E_{\uparrow} - \frac{\mathcal{Q}_{\downarrow}}{r^2} - i\partial_r]f_{\uparrow} &= m h_{\uparrow}, \\ [E_{\uparrow} - \frac{\mathcal{Q}_{\downarrow}}{r^2} + i\partial_r]h_{\uparrow} &= m f_{\uparrow}, \end{aligned} \quad (5.16)$$

so that now the singular central potential becomes attractive. There is no need to repeat all the calculations because the change of the sign of  $\mathcal{Q}$  is compensated by the change of the signs of  $E$  and  $m$ . The difference is that now the bound states of the upper continuum will tend to compensate the “external” charge so that there will be a different number of bound states. In order to guarantee the positive energy of the vector fields, the coupling constants of the electron were chosen negative. If one of the localized states with positive energy is associated with the electron, then the localized states at negative energies will interact with the electron as positive charges. Being close to the original Dirac idea, this picture does not require a continuous sea of occupied states. The trends of this oversimplified model clearly point to the existence of some optimal self-consistent values of  $m$  and  $\mathcal{Q}$ , which should be looked for as the eigenvalues of the original system of equations (3.9) and (3.11).

To this point, the picture of “falling onto a centre” renders the probability of finding a Dirac field at this centre finite. The distribution of probability density is amazingly uniform within the range  $0 < r < r_{max} \sim \mathcal{Q} \sim 1/m$ , so that a true singularity at the origin is not

likely to develop. However, it is the peak of probability density  $R(0)$ , which determines the magnitude of a gravitational mass and makes it proportional to the sharpness of the localization of the Dirac field. The smallest particles are the heaviest, and we indeed have  $R(0) \sim |E|$ . This is evidence that, even being of a different physical origin, the inertial ( $\sim \int T^{00} dV \sim E$ ) and gravitational ( $\sim m \int \mathcal{P} dV \sim R(0)$ ) masses of stable particles are the same, up to a possible factor which can be absorbed into a “Newton’s gravitational constant”  $G_N \sim (g^2/M^2)$ . Only the full solution of Eqs. (3.9) and (3.11) can tell if this factor is universal. An affirmative answer will be the ultimate proof that inertial and gravitational masses are equal. It looks like the mass, which corresponds to this gravitational constant, is smaller than the formal “dimensional” Planck mass,  $M_{Planck}^2 = (\hbar c/G_N)$ , by a factor  $g$ , related to the electro-weak interactions. We suggest that the electro-weak and gravitational constants uniquely determine the mass  $M$  of the axial field.

Finally, it is not clear yet if the localized solutions of the toy model of this section match by their properties any presently known form of matter, or even if they are stable. (As it was mentioned in Sec. III, it is not obvious that the static solutions of sine-Gordon even exist.) Being strongly localized, such solutions are not likely to effectively interact with the “normal” matter. Nevertheless, these spherically-symmetric solutions clearly are a source of the gravitational field and should be considered as the candidates for the dark matter.

## VI. CONCLUDING REMARKS.

The axial field  $\aleph_a$  seems to have been known in connection with various physical phenomena and under various names for a long time. The divergence of this field is proportional to the divergence of the axial current which carries the quantum numbers of a pion field. In the non-relativistic limit, the axial field in the Pauli equation [1] adequately describes the parity nonconservation phenomena in atomic physics. Together with the Maxwell field in the spin connection it can reproduce the effects of neutral and charged currents of the standard model provided, as it was conjectured in Sec.V, that the observed sign of the electric charge is connected with the sign of energy and polarization of the localized Dirac states. In other words, the immediately anticipated physical effects of the axial field seem to be known or expected.

The new physical *quality* is that, like the Maxwell field, the axial field comes into sight as one of the ingredients of parallel transport of the Dirac field. It is a kinematic effect stemming from the complex nature of the spinor field and a large number of its polarization degrees of freedom. Truly surprising is the fact that this field is derived at the most basic level of Lorentz invariance and that it was not motivated by any phenomenological input, e.g., by a need to justify an observed conservation law.

The new result is that the existence of the axial field and an additional guess that the Dirac field *can* form compact objects (particles) is enough to initiate an effective mechanism of the auto-localization of the Dirac field. Indeed, *it would be extremely difficult to find any counterarguments to this guess in a real world.* If this mechanism is actually realized in Nature, then all the localized matter in the Universe can be viewed as a collection of various clusters of the Dirac field, which are tightly bound by the axial field at the shortest distances and weakly interact via the same field at the largest ones. Furthermore, the existence of a so natural mechanism of auto-localization allows one to think that the known symmetries and conservation laws can be a consequence of the soliton nature of the Dirac field.

The nonlinear equations for the Dirac field were advocated long ago by Heisenberg [17] and by Thirring (massless Thirring model in 1+1 dimension, [18]). The correspondence between the quantum sine-Gordon equation and the massive Thirring model was established by Coleman [19].

The equations of Sec.III allow one to derive a phenomenological Lagrangian of a theory that apparently includes the *axion* field [20]. The field  $\Upsilon$  and the axion field couple to the Dirac field in the same way and the effective Lagrangian includes terms that lead to the sine-Gordon type equation of motion for the axion field. However, one cannot associate  $\Upsilon$  with a spinless particle because it is a descendent of a vector field  $N_\mu$  in the classical limit.

It was not expected *a priori* that in order that the Dirac field could form compact objects the axial field must be the gravitational field. However, the picture of gravity as an effect of the axial field clearly matches all known properties of this interaction. In addition to equations (4.8) and (4.4) (the Einstein field equation and the local equivalence of the forces of gravity and of inertia) this picture explains gravity as a coherent effect that cannot be screened by any bodies or fields. The impossibility to screen gravity seems to be a consequence of the singular nature of the gravitational potential. The “fall onto a centre” is universal and is guaranteed by a special position of the Newton’s force as a potential in the Dirac equation. Einstein’s field equations, even without the energy momentum tensor of a matter, correctly match the motion of material bodies as clusters of the Dirac field. It would be fair to say that the Dirac field is so natural “stuffing” for the Einstein’s singularities, that the previous feeling of the incompleteness of the Einstein-Infeld theory fades away. Of all possible solutions of the Einstein equations the physical meaningful are those that have the realistic partners among the solutions of Dirac equation with the axial field.

An advantage of the new theory is the small number of initial assumptions and parameters; it exploits only the fact that spinors realize a representation of a local

group of Lorentz transformations. The hitherto explored consequences of this approach seem to bring us closer to the explanation of several, yet not perfectly understood facts:

(i) The Sommerfeld formula of fine structure, which treats the electron as a point-like charge, nevertheless, reproduces the exact answer of the Dirac theory. Auto-localization of Dirac field naturally integrates an image of a classical particle into the full scope of field theory and demystify this coincidence.

It also eliminates a difficult-to-justify procedure of “switching on” the interaction in the S-matrix version of field theory (see, e.g. [21]), because the colliding localized particles indeed begin to strongly interact only after they physically overlap [22].

The auto-localization totally removes the painful problem of collinear divergences in radiative corrections, which frustrates calculations of perturbation theory in the limit of massless charged particles. Such a limit just does not exist in the real world.

(ii) In Sec.V of paper [1] it was pointed out that the axial field has an effective, with respect to its action on atomic systems, magnetic equivalent  $\vec{B}_N = (g/\mu_B) \vec{N}$ . If  $\vec{N}$  is the axial-Newton field of a galaxy, then it may serve as a seed field from which the effect of the galactic dynamo takes over [23]. Even though this field is very small, there is enough time to accumulate the effect of magnetic-type polarization or other parity-odd effects.

The theory probably offers a new insight into such questions as, what the amount of *gravitating matter* in the Universe is, and what form the matter at the latest stages of stellar evolution and at gravitational collapse takes.

(iii) V. Gribov [24] advocated the “falling onto a centre” phenomenon as a mechanism of quark and color confinement. At short distances, the axial field behaves adequately for this hypothesis and allows one to explore it using much more economic means.

A nearly singular axial potential makes the phase shift between incoming and outgoing waves of any scattering process (which is not a hidden parameter) uncertain. One can wonder if this can be a reason why multiply repeated collisions form a statistical ensemble. Furthermore, it is not so obvious that equation (3.11) for the singular part of axial potential has static solutions at all. If so, then any interaction in real matter is affected by this invisible from outside temporal dynamics in compact spinor configurations.

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- [25] In his paper [2], Fock emphasized the significance of the presence of the Ricci tensor side by side with the field strength tensor  $F_{\mu\nu}$ , deferring any further discussion of this fact.
- [26] This very much resembles general relativity: the affine connection  $\Gamma_{\mu\nu}^\lambda$  has, in general,  $4^3 = 64$  components. However, if it defines a parallel transport of four vector fields, e.g. a tetrad, then its 64 components can be expressed via only 16 functions.
- [27] If (3.0) is not perfectly satisfied, then one may *think of* a not perfectly stable cluster where the field  $N_\mu$  is not an exact gradient of  $\Upsilon(x)$ , and where some remnant of the field strength  $U^{\sigma\mu}$  may interfere with  $F^{\sigma\mu}$ . Most likely, this will lead to the loss of the normalizable states, etc. However, this seems to be a natural way to discover an entire spectrum of *unstable* elementary particles as the excitations of the Dirac field.
- [28] Since  $\mathcal{R}^2 = j^a j_a$ , we have  $\mathcal{R} = 0$  (and must call this space empty) when the density of probability current in it is a light-like vector. In this case, the arguments that have led to the ansatz (3.0) are not valid. The two-component spinor field cannot form a compact object. It cannot be a source of the field  $\Upsilon$ .